MATH 2443
1st Midterm Review Solutions

1. Let

$$f(x) = \begin{cases} \sqrt{x^2 - y}, & (x,y) \neq (0,-5),(2,-5) \\ 3, & (x,y) = (0,-5),(2,-5) \end{cases} .$$

   (a) Find and sketch the domain of f.

   The domain is $D = \{(x,y) | y \leq x^2\}$. The graph on the xy-plane is the parabola $y = x^2$ with the region below the parabola shaded.

   (b) Find the range of f.

   The range is $R = [0, \infty)$.

   (c) Where is f continuous?

   The function $\sqrt{x^2 - y}$ is continuous on its domain so f will be continuous on its domain except possibly at the points $(0,-5)$ and $(2,-5)$. At $(0,-5)$ we have that

$$\lim_{(x,y)\to(0,-5)} f(x,y) = \sqrt{5} \neq f(0,-5)$$

   so f is not continuous here. At $(2,-5)$, we have that

$$\lim_{(x,y)\to(2,-5)} f(x,y) = 3 = f(2,-5)$$

   so f is continuous here.

   So f is continuous on the points $\{(x,y)|y \leq x^2, (x,y) \neq (0,-5)\}$.

2. Find the limit or show it does not exist.

   (a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3y + xy^3}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{xy(x^2 + y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} xy = 0$$

   (b)

$$\lim_{(x,y)\to(0,0)} \frac{x^3y + xy^3}{x^4 + y^4}$$

   This limit does not exist. Approaching on the line $y = x$ gives $\lim_{x\to 0} \frac{x^4+x^4}{x^4+x^4} = 1$ while approaching on the line $x = 0$ gives $\lim_{y\to 0} \frac{0}{y^4} = 0$.

   (c)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{1 - e^{x^2+y^2}}$$

Changing to polar we get that this limit is equal to

$$\lim_{r\to 0} \frac{r^3\cos^2(\theta)\sin(\theta)}{1-e^{r^2}}$$

then applying L'Hopital's rule, this is equal to

$$\lim_{r\to 0} \frac{3r^2\cos^2(\theta)\sin(\theta)}{-2re^{r^2}} = \lim_{r\to 0} \frac{3r\cos^2(\theta)\sin(\theta)}{-2e^{r^2}} = \frac{0}{-2} = 0\ .$$

3. I have two functions, $f(x, y)$ and $g(x, y)$. I compute f_x and g_x and get two of these three functions: $e^y, 2xy, x + y$. Then I compute f_y and g_y and get two of these three functions: $2x + e^y, xe^y, x^2 + e^y$. Which function in the first list is not one of f_x or g_x and which function in the second list is not f_y or g_y?

   Taking the derivative with respect to y of the first three functions tells us that f_{xy} and g_{xy} are two of the three functions $e^y, 2x, 1$. Similarly, taking the derivative with respect to x of the second three functions tells us that f_{yx} and g_{yx} are two of the three functions $2, e^y, 2x$. Then as $f_{xy} = f_{yx}$ and $g_{xy} = g_{yx}$ we see that the f_{xy} and g_{xy} must be e^y and $2x$.

   In the first list, $x + y$ is not f_x or g_x. In the second, $2x + e^y$ is not f_y or g_y.

4. The equation $x^2yz - z^2 = x^3 - y^2$ determines a surface through the point $(1, 2, 3)$.

   (a) Find the equation of the tangent plane to the surface at this point.
   Rewriting the equation as $x^2yz - z^2 - x^3 + y^2 = 0$ and taking $f(x, y, z) = x^2yz - z^2 - x^3 + y^2$ we have that the tangent plane at $(1, 2, 3)$ will have formula

   $$f_x(1, 2, 3)(x - 1) + f_y(1, 2, 3)(y - 2) + f_z(1, 2, 3)(z - 3) = 0\ .$$

   Then $f_x(x, y, z) = 2xyz - 3x^2$ so $f_x(1, 2, 3) = 9$, $f_y(x, y, z) = x^2z + 2y$ so $f_y(1, 2, 3) = 7$, and $f_z(x, y, z) = x^2y - 2z$ so $f_z(1, 2, 3) = -4$.
   This gives the tangent plane equation $9(x - 1) + 7(y - 2) - 4(z - 3) = 0$

   (b) Viewing z as an implicitly defined function of x and y near the point $(1, 2, 3)$, compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x = 1$, $y = 2$.
   Using implicit differentiation with respect to x we get that $2xyz + x^2y\frac{\partial z}{\partial x} - 2z\frac{\partial z}{\partial x} = 3x^2$. Solving for the derivative gives $\frac{\partial z}{\partial x} = \frac{3x^2-2xyz}{x^2y-2z}$ which is $9/4$ at $(1, 2, 3)$.
   Using implicit differentiation with respect to y we get that $x^2z + x^2y\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = -2y$. Solving for the derivative gives $\frac{\partial z}{\partial y} = \frac{-2y-x^2z}{x^2y-2z}$ which is $7/4$ at $(1, 2, 3)$.

5. Given a function $z = f(x, y)$, one of the following three functions is f_x and one is f_y. Identify them. The functions are: $e^x + e^y, e^x + y, xe^y + y$.

   We know that f_x is one of the three functions above so f_{xy} is one of $e^y, 1, xe^y + 1$. On the other hand f_y is one of the three functions so $f_{xy} = f_{yx}$ is one of e^x, e^x, e^y. Thus f_{xy} is e^y so $f_x = e^x + e^y, f_y = xe^y + y$.

6. Let $z = x\cos(y^2) + e^{xy}$. Use differentials to estimate the change in z as x changes from 7 to 7.1 and y changes from 0 to -.1.

   The formula for the differential is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. The partial derivatives are $\frac{\partial z}{\partial x} = \cos(y^2) + ye^{xy}$ and $\frac{\partial z}{\partial y} = -2xy\sin(y^2) + xe^{xy}$. At $(7, 0)$ these are $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 7$. Also $dx = .1$ and $dy = -.1$, so $dz = 1(.1) + 7(-.1) = -.6$.

7. The two shorter sides of a right triangle are measured then used to calculate the length of the hypotenuse. The error in measurement of the sides is at most 1 %. Use differentials to estimate the maximum percent error in the length of the hypotenuse.

   Let x, y be the short sides of the triangle. Then the length of the hypotenuse is $z = f(x, y) = \sqrt{x^2 + y^2}$. The differential dz is given by $dz = f_x(x, y)dx + f_y(x, y)dy$. There is a possible 1 % error in x and y so $dx = .01x$ and $dy = .01y$. Then $f_x(x, y) = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y(x, y) = \frac{y}{\sqrt{x^2+y^2}}$ so

$$dz = \frac{x}{\sqrt{x^2 + y^2}}(.01x) + \frac{y}{\sqrt{x^2 + y^2}}(.01y) = \frac{(.01)(x^2 + y^2)}{\sqrt{x^2 + y^2}} = (.01)\sqrt{x^2 + y^2}\ .$$

   Divide by z to get the percent error

$$\frac{dz}{z} = \frac{(.01)\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = .01$$

   so the percent error is 1 %.

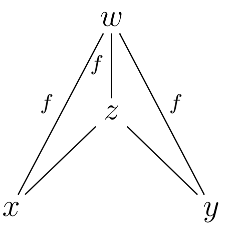
8. Suppose that $w = f(x, y, z)$ is a differentiable function and that $w = 4$ when $x = 1, y = 2, z = 3$. If $f_x(1, 2, 3) = 5, f_y(1, 2, 3) = -1, f_z(1, 2, 3) = 2$, compute a reasonable approximation for $f(.9, 2.1, 3.2)$.

   Suppose now that x, y and z are not really independent and that as x and y vary, z is constrained to move so that $xyz = 6$. As a result, we can view w as a function of x and y and we write $w = h(x, y)$ to denote this function. Compute $h_y(1, 2)$.

   We will first approximate $f(.9, 2.1, 3.2)$ using the linearization $f(x, y, z) \approx f(1, 2, 3) + f_x(1, 2, 3)(x - 1) + f_y(1, 2, 3)(y - 2) + f_z(1, 2, 3)(z - 3) =$

$4 + 5(x-1) - (y-2) + 2(z-3)$. So
$f(.9, 2.1, 3.2) \approx 4 + 5(-.1) - (.1) + 2(.2) = 3.8$.

To find $h_y(1,2)$ we draw the following tree diagram.

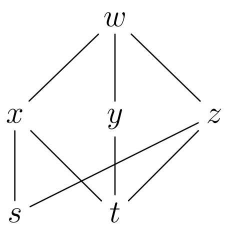


We use the chain rule to get that $h_y(1,2) = f_y(1,2,3) + f_z(1,2,3)\frac{\partial z}{\partial y}$. Using that $z = 6/(xy)$ we get that $\frac{\partial z}{\partial y} = -6/(xy^2)$ which is $-3/2$ at $x = 1, y = 2$ so $h_y(1,2) = -1 + 2(-3/2) = -4$.

9. Find $\frac{\partial w}{\partial s}$ when $s = 1$ and $t = 1$ where

$$w = f(x,y,z) = x^2 + (y\sqrt{5 + \arctan z})\frac{e^{z^3 - \sqrt{y^4+z}}}{\ln(3 + \cos(\sin(z) + y))}$$

and $x = s^2 + st + t^2$, $y = t^3$, $z = 2st - s^2$.

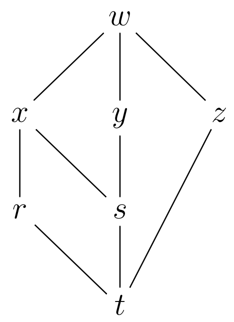


We can draw a tree diagram to get that $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$. We want to avoid computing $\frac{\partial w}{\partial z}$ if possible so we compute the easy derivatives first:
$\frac{\partial w}{\partial x} = 2x$, $\frac{\partial x}{\partial s} = 2s + t$, $\frac{\partial z}{\partial s} = 2t - 2s$. At $t = 1, s = 1$ we have $x = 3$ and so $\frac{\partial w}{\partial x} = 6$, $\frac{\partial x}{\partial s} = 3$, $\frac{\partial z}{\partial s} = 0$. Then

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 6 \cdot 3 + \frac{\partial w}{\partial y} \cdot 0 = 18 \ .$$

Note that we did not need to find $\frac{\partial w}{\partial z}$.

10. Suppose $w = xy^2 + zx^2$, $x = rs$, $y = s^2$, $z = t^4$, $s = 2t$, $r = e^{t-1}$. Draw a tree diagram and find $\frac{dw}{dt}$ when $t = 1$.



Using the chain rule, we get that

$$\frac{dw}{dt} = \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial y}\frac{dy}{ds}\frac{ds}{dt} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}\frac{ds}{dt} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial r}\frac{dr}{dt} \ .$$

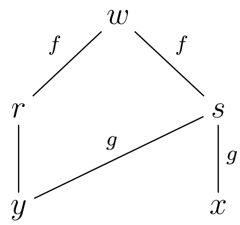
When $t = 1$ the other variables are $r = 1, s = 2, z = 1, y = 4, x = 2$ and the partial derivatives are: $\frac{\partial w}{\partial z} = x^2 = 4$, $\frac{dz}{dt} = 4t^3 = 4$, $\frac{\partial w}{\partial y} = 2xy = 16$, $\frac{dy}{ds} = 2s = 4$, $\frac{ds}{dt} = 2$, $\frac{\partial w}{\partial x} = y^2 + 2xz = 20$, $\frac{\partial x}{\partial s} = r = 1$, $\frac{\partial x}{\partial r} = s = 2$, and $\frac{dr}{dt} = e^{t-1} = 1$.

Combining these results we get that

$$\frac{dw}{dt} = 4 \cdot 4 + 16 \cdot 4 \cdot 2 + 20 \cdot 1 \cdot 2 + 20 \cdot 2 \cdot 1 = 224 \ .$$

11. Given functions $f(r, s)$ and $g(x, y)$, create a new function by the formula $w = f(y^2, g(x, y))$. Using the following data, compute the values of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ when $x = 1, y = 2$. (Assume that all partial derivatives are continuous). $g(1, 2) = 3$, $g_x(1, 2) = 4$, $g_y(1, 2) = 5$, $f(4, 3) = 6$, $f_r(4, 3) = 7$, $f_s(4, 3) = 2$, $f(1, 2) = 1$, $f_r(1, 2) = 3$, $f_s(1, 2) = 9$.

    Rewrite this as $w = f(r, s)$ where $r = y^2, s = g(x, y)$. Then we get the following tree diagram.



When $x = 1$ and $y = 2$ we have that $r = 4, s = g(1, 2) = 3$. Then

$$\frac{\partial w}{\partial x} = f_s(4, 3)g_x(1, 2) = 2 \cdot 4 = 8$$

and

$$\frac{\partial w}{\partial y} = f_s(4, 3)g_y(1, 2) + f_r(4, 3)\frac{dr}{dy} = 2 \cdot 5 + 7(2y) = 10 + 7 \cdot 4 = 38 \ .$$

12. Let $f(x, y) = x^2y^2 + axy - y^4$ where a is some constant. The directional derivative of f at the point $(1, 1)$ in the direction of the point $(5, 4)$ is -1.

   (a) Find a.

   The partial derivatives are $f_x(x, y) = 2xy^2 + ay$, $f_y(x, y) = 2x^2y + ax - 4y^3$ so at $(1, 1)$ they are $f_x(1, 1) = 2 + a$, $f_y(1, 1) = -2 + a$ and the gradient is $\nabla f(1, 1) = \langle 2 + a, -2 + a\rangle$. Moving from $(1, 1)$ to $(5, 4)$ is the vector $\langle 4, 3\rangle$ which has magnitude 5 so the unit vector in that direction is $u = \langle 4/5, 3/5\rangle$. This gives a direction derivative of

   $$D_u f(1, 1) = \nabla f(1, 1) \cdot u = \langle 2 + a, -2 + a\rangle \cdot \langle 4/5, 3/5\rangle = (7a + 2)/5 \ .$$

   Setting this equal to -1 and solving for a gives that $a = -1$.

   (b) What are the maximum and minimum values of the directional derivative of f at the point $(1, 1)$?

   The maximum value of the directional derivative is $|\nabla f(1, 1)| = |\langle 1, -3\rangle| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$. The minimum value of the directional derivative is $-|\nabla f(1, 1)| = -\sqrt{10}$.

   (c) Find a point such that the directional derivative at $(1, 1)$ in the direction of that point is as small as possible.

   The directional derivative will be as small as possible if we move in the direction of $-\nabla f(1, 1) = \langle -1, 3\rangle$. Any point in this direction is correct, perhaps the easiest one to find is $(1 - 1, 1 + 3) = (0, 4)$.

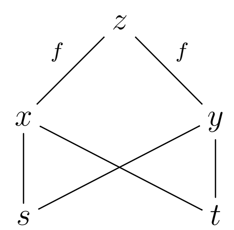
13. Suppose that the function $f(x, y)$ is differentiable and assume that $f(3, 4) = 7$ and $\nabla f(3, 4) = \langle 3, -2\rangle$.

   (a) Find a reasonable approximation for $f(2.8, 4.1)$.

   $f(2.8, 4.1) \approx f(3, 4) + f_x(3, 4)(2.8 - 3) + f_y(3, 4)(4.1 - 4) =$ $7 + 3(-.2) + (-2)(.1) = 6.2$, where the values of the partial derivatives come from the gradient $\nabla f(3, 4)$.

   (b) Let $h(s, t) = f(s^2 + t, st + 2s)$. Compute $h_s(1, 2)$.

   Let $z = h(s, t)$. Then $z = f(x, y)$ where $x = s^2 + t, y = st + 2s$ so we get the following tree diagram.



Then $h_s(1,2)$ is $\frac{\partial z}{\partial s}$ when $s = 1, t = 2$. This gives us that

$$h_s(s,t) = f_x(x,y)\frac{\partial x}{\partial s} + f_y(x,y)\frac{\partial y}{\partial s} = f_x(x,y)(2s) + f_y(x,y)(t+2)\ .$$

When $s = 1$ and $t = 2$ we get that $x = 3, y = 4$ so

$$h_s(1,2) = f_x(3,4) \cdot 2 + f_y(3,4) \cdot 4 = 3(2) + (-2)(4) = -2.$$

(c) Now suppose that $g(t)$ is a function that has the property that $f(g(t), t^2) = 7$ for all values of t. If $g(2) = 3$, compute $g'(2)$.
Let $z = f(x,y)$ where $x = g(t), y = t^2$. Then $\frac{dz}{dt} = 0$ because $z = 7$ for any t. Using the chain rule, we get that
$0 = \frac{dz}{dt} = f_x(x,y)g'(t) + f_y(x,y)(2t)$. In particular, if $t = 2$ then
$0 = f_x(g(2),4)g'(2) + f_y(g(2),4) \cdot 4 = f_x(3,4)g'(2) + 4f_y(3,4) = 3g'(2) - 8$.
Solving $0 = 3g'(2) - 8$ gives us that $g'(2) = 8/3$.

14. Let $z = x(e^y + x)$.

(a) Compute $\partial z/\partial x$, $\partial z/\partial y$, and $\partial^2 z/\partial y^2$.
$\partial z/\partial x = e^y + 2x$, $\partial z/\partial y = xe^y$, and $\partial^2 z/\partial y^2 = xe^y$.

(b) Find ∇z at the point $(2,0)$.
At $(2,0)$, $\partial z/\partial x = e^y + 2x = 5$ and $\partial z/\partial y = xe^y = 2$ so $\nabla z(2,0) = \langle 5, 2\rangle$.

(c) What is the directional derivative of z at the point $(2,0)$ in the direction toward $(-1,4)$.
The vector from $(2,0)$ to $(-1,4)$ is $\langle -3, 4\rangle$ which has magnitude 5, so the unit vector in the direction of $(-1,4)$ is $u = \langle -3/5, 4/5\rangle$. Then
$D_u z(2,0) = \nabla z(2,0) \cdot u = \langle 5,2\rangle \cdot \langle -3/5, 4/5\rangle = -7/5$.

15. Suppose f is a differentiable function and at the point $(17,-23)$ the directional derivative of f in the direction of the vector $\langle 3,-1\rangle$ is $-\frac{11}{\sqrt{10}}$. At that same point, the directional derivative of f in the direction of $\langle 2,7\rangle$ is $\frac{31}{\sqrt{53}}$. Find the directional derivative of f at $(17,23)$ in the direction of $\langle -2,1\rangle$.

Write $\nabla f(17,-23) = \langle a,b\rangle$. The unit vector in the direction of $\langle 3,-1\rangle$ is $\langle 3/\sqrt{10}, -1/\sqrt{10}\rangle$ so

$$-\frac{11}{\sqrt{10}} = D_u f(17,23) = \langle a,b\rangle \cdot \langle 3/\sqrt{10}, -1/\sqrt{10}\rangle = (3a-b)/\sqrt{10}\ .$$

This gives us the equation $3a - b = -11$. Similarly, the unit vector in the direction of $\langle 2,7\rangle$ is $\langle 2/\sqrt{53}, 7/\sqrt{53}\rangle$ so

$$\frac{31}{\sqrt{53}} = D_u f(17,23) = \langle a,b\rangle \cdot \langle 2/\sqrt{53}, 7/\sqrt{53}\rangle = (2a+7b)/\sqrt{53}$$

which gives us the equation $2a + 7b = 31$. Solving the system of equations $3a - b = -11, 2a + 7b = 31$ for a and b gives $a = -2, b = 5$ so $\nabla f(17, -23) = \langle -2, 5\rangle$. Then unit vector in the direction of $\langle -2, 1\rangle$ is $\langle -2/\sqrt{5}, 1/\sqrt{5}\rangle$ so

$$D_u f(17, 23) = \langle -2, 5\rangle \cdot \langle -2/\sqrt{5}, 1/\sqrt{5}\rangle = 9/\sqrt{5}\ .$$

16. Find the point or points on the curve $2y^3 + 9x^2 = 16$ that are closest to the origin.

    We want to minimize the distance from (x, y) to $(0, 0)$ which is $\sqrt{x^2 + y^2}$. The minimum of $\sqrt{x^2 + y^2}$ occurs at the same place as the minimum of $x^2 + y^2$ so we can instead find where $f(x, y) = x^2 + y^2$ is minimal under the constraint $g(x, y) = 2y^3 + 9x^2 = 16$. Use Lagrange multipliers to get that $\langle 2x, 2y\rangle = \nabla f = \lambda\nabla g = \lambda\langle 18x, 6y^2\rangle$. We get the system of equations: $2x = \lambda 18x, 2y = \lambda 6y^2, 2y^3 + 9x^2 = 16$. The first equation tells us that $x = 0$ or $\lambda = 1/9$. If $x = 0$ then the third equation tells us that $y = 2$ and we get the critical point $(0, 2)$. If $\lambda = 1/9$ then the second equation becomes $2y = (6/9)y^2$ which has solutions $y = 0, 3$. If $y = 0$ then the third equation gives that $x = 4/3, -4/3$ and we get the critical points $(4/3, 0), (-4/3, 0)$. Plugging in $y = 3$ to the third equation we see that this is not possible so there are no more critical points.

    The values of f at each critical point are $f(0, 2) = 4, f(\pm 4/3, 0) = 16/9$. The curve is the function $y = \sqrt[3]{(16 - 9x^2)/2}$ so the segment from where $x = -2$ to where $x = 2$ is a closed and bounded region and the value of f at the endpoints of this region is at least 4 so we see that on this region the absolute min occurs at $(4/3, 0), (-4/3, 0)$. On the rest of the curve, we have that $|x| > 2$ so $f(x, y) > 4$ and thus these points are where the absolute min occurs on the entire curve.

17. The function $w = x^2 + y - xy$ is defined on the region bounded by the curve $y = 9 - x^2$ and the x-axis. Find the maximum and minimum values of w on this region and the points where they occur.

    This is a closed and bounded region so we can find the absolute max and min by finding all critical points and the value of w at each one. Write $f(x, y) = x^2 + y - xy$. First check for interior critical points. The partial derivatives are $f_x = 2x - y, f_y = 1 - x$. These are both 0 when $x = 1, y = 2$ so we get the critical point $(1, 2)$ and this is a point inside the region. Next check for critical points on $y = 9 - x^2$. This can be done either by plugging $y = 9 - x^2$ into f and finding where the derivatives of this function are 0, or with Lagrange multipliers. Using Lagrange multipliers with constraint $g(x, y) = y + x^2 = 9$ we get the equations $2x - y = \lambda 2x, 1 - x = \lambda, x^2 + y = 9$. The first equation can be rearranged as $y = 2x(1 - \lambda)$ and the second equation as $x = 1 - \lambda$. Combining these we get $y = 2x^2$ and plugging this into $y + x^2 = 9$ we get the critical points $(\sqrt{3}, 6), (-\sqrt{3}, 6)$. On the x-axis, $y = 0$ so $f(x, y) = f(x, 0) = x^2$ which has derivative $2x$ and thus is critical at $x = 0$ so we get the critical point $(0, 0)$. Finally, we all need to check the corner points of our region where $y = 9 - x^2$ and $y = 0$ meet which are $(3, 0)$ and $(-3, 0)$. We thus have the following critical points: $(1, 2)$, $(\sqrt{3}, 6)$, $(-\sqrt{3}, 6)$, $(0, 0)$, $(3, 0)$, $(-3, 0)$. The values of w at these points respectively are $1, 9 - 6\sqrt{3}, 9 + \sqrt{3}, 0, 9, 9$. The maximum is $9 + 6\sqrt{3}$ at the point $(-\sqrt{3}, 6)$ and the minimum is $9 - 6\sqrt{3}$ at $(\sqrt{3}, 6)$.

18. Find the minimum of $w = x^2 + 2y^2 + 3z^2$ on the plane $x + y + z = 1$ and where it occurs.

    Using Lagrange multipliers gives the system of equations $2x = \lambda$, $4y = \lambda$, $6z = \lambda$, and $x + y + z = 1$. Then $2x = 4y$ so $y = x/2$. Similarly $2x = 6z$ so $z = x/3$. Plugging in to $x + y + z = 1$ gives $x + x/2 + x/3 = 1$ so $x = 6/11$. Then $y = x/2 = 3/11$ and $z = x/3 = 2/11$. Thus we have one critical point at $(6/11, 3/11, 2/11)$ and the value of w at this point is $6/11$.

    This points must be an absolute minimum. To see this, consider the region which is the intersection of $x + y + z = 1$ and $x^2 + 2y^2 + 3z^2 \leq 1$. This is the intersection of a solid ellipsoid with a plane so it is a closed and bounded region and w must have an absolute minimum on this region. The only interior critical point is $(6/11, 3/11, 2/11)$ where $w = 6/11$ and everywhere on the boundary $w = 1$. Thus the absolute minimum of w on this region is $6/11$. Outside of this region on the plane $x + y + z = 1$ we must have that $w > 1$ so $6/11$ is the absolute minimum for the whole plane.

19. A solid spherical ball of radius 3 is centered at the origin. The temperature at the point (x, y, z) is given by $T(x, y, z) = 4x + 2y + z^2$. Find the maximum and minimum temperatures on the ball and where they occur.

    This is a closed and bounded region so to find the absolute max and min, we just need to find all the critical points and see which one has the largest value of T and which has the smallest value of T. First check for critical points on the interior of the ball. Any interior critical point will be where all the partial derivatives of T are 0. But $T_x = 4$ is never 0 so there are no interior critical points. Next use Lagrange multipliers to find critical points on the boundary of the ball, $x^2 + y^2 + z^2 = 9$. We get the system of equations $4 = \lambda 2x$, $2 = \lambda 2y$, $2z = \lambda 2z$, and $x^2 + y^2 + z^2 = 9$. The third equation $2z = \lambda 2z$ implies that $\lambda = 1$ or $z = 0$. First suppose $\lambda = 1$. Then $4 = 2x$ so $x = 2$ and $2 = 2y$ so $y = 1$. Then $2^2 + 1^2 + z^2 = 9$ so $z = \pm 2$. So we get the critical points $(2, 1, 2)$ and $(2, 1, -2)$. Next suppose that $z = 0$. Using the first two equations from the system of equations, we get that $4y = 2xy\lambda = 2x$ so $x = 2y$. Plug in $z = 0$ and $x = 2y$ to $x^2 + y^2 + z^2 = 9$ to get $4y^2 + y^2 = 9$ so $y = \pm \frac{3}{\sqrt{5}}$ so we get the critical points $(\frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}}, 0)$ and $(-\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}, 0)$. So we have a total of 4 critical points. Evaluate T at these points: $T(2, 1, 2) = 14$ and $T(2, 1, -2) = 14$, $T(\frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}}, 0) = 6\sqrt{5}$, and $T(-\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}, 0) = -6\sqrt{5}$. So the max is 14 at $(2, 1, \pm 2)$ and min is $-6\sqrt{5}$ at $(-\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}, 0)$.

20. Let $f(x, y) = 2x^2y + \frac{1}{2}y^2 - x^4 - 12y$. Find all critical points of f. For each critical point, determine if it is a local max, a local min, or a saddle point.

    The partial derivatives are $f_x = 4xy - 4x^3, f_y = 2x^2 + y - 12$. These functions exist everywhere, so the critical points will be where both equations are 0. The first equation is 0 if $x = 0$ or $y = x^2$. If $x = 0$ then $2x^2 + y - 12 = 0$ implies $y = 12$ so we get the critical point $(0, 12)$. If $y = x^2$ then the second equation becomes $2x^2 + x^2 = 12$ so $x = \pm 2$ and $y = 4$. This gives us the critical points $(2, 4), (-2, 4)$.

    Use the second derivative test to determine what is happening at each critical point. The second order partial derivatives are $f_{xx} = 4y - 12x^2, f_{yy} = 1, f_{xy} = 4x$. Let $D = f_{xx}f_{yy} - (f_{xy})^2$. Then at $(0, 12)$ we have that $D = 48 \cdot 1 - 0^2 = 48$. Thus $D > 0$ and $f_{xx} = 48 > 0$ so f has a local minimum at $(0, 12)$. At $(2, \pm 4)$ we have that $D = (-32) \cdot 1 - (\pm 8)^2 = -96$ so $D < 0$ and f has saddle points at $(2, 4)$ and $(-2, 4)$.